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Pedagogical Study on the Meaning of Mathematical Structure

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This thesis examines how Bourbaki's matrix structures and Piaget's structure of mathematical thought are related on the basis of Piaget's study on genetic epistemology and structuralism. In addition, it reviews the essential characteristics that the structure of knowledge should have and also points out that transformation and development are the key factors of the structure. In the light of this, it indicates the limits of Bruner's concept of structure and discusses about having the learner construct the algebraic structure as a mathematical concept.

Key Words : Structuralism, Genetic epistemology, Structure

When constructing and carrying out mathematics curricula, we first encounter an important question: "What shall we teach?" As an answer to this question, Bruner suggests in his book, The Process of Education(1960), that the structure of mathematics should be taught. This idea gains much support in the education world because it is accepted as an alternative to the existing education system where only piles of facts or "middle language" is being taught.

What does teaching mathematical structures to students mean? If a math teacher teaches high school students commutative law, associative law and the existence of inverse element and identity element as the properties of the real number system, is he teaching the algebraic structure? For students, who do not know the concepts like group, ring and field and also have not noticed the similar phenomena in a set of polynomials, a set of functions and a set of matrices, the algebraic structure presented as above is likely to be a skeletonlike structure extremely poor in context. In other words, if students, who have already gotten familiar with the four basic operations of real numbers through middle school curriculum, accept the algebraic structure as nothing new to learn or think that the unfamiliar terms such as inverse element and identity element should be just memorized, we have to reconsider the present efforts to teach the structure of mathematics.

The focus of this study is on considering the meaning of the structure that we are trying to teach. Describing the axiomatic systems for algebra and having student memorize them is clearly not equal to teaching the algebraic structures. Nevertheless, there seem to be no other ways to present "structure" except just describing the axiomatic systems. Regarding this problem, Yim(1998) stresses that teachers should make a thorough study of teaching materials and broaden their knowledge by themselves in order to teach the structures hidden behind the axiomatic systems. However, teachers' background knowledge or preparation for teaching is not a sufficient condition for teaching but only a necessary condition. That is to say, even if a teacher had a wide knowledge of the algebraic structures like group, ring and field, with the system and the order of the existing curricular he could not but describe just the axiomatic systems for algebra to his students, promising that "what is being taught now will be made a more universal and significant structure by what will be taught later". Therefore, to solve this problem, we first have to examine the subject of the genesis of structure which has been overlooked so far in the discussions on the structure of mathematical knowledge.

Structure of knowledge and cognitive structure

How is human knowledge made up? How do human beings know things? These may be the most fundamental topics of epistemology. And if education can be referred to as teaching knowledge, these questions about cognition are essential in the field of education. Plato raised such questions through "Meno's paradox" already two thousand years ago. Let's say there is a man who absolutely knows nothing. Then, how can he come to know something that he has not known before? This problem is not simple at all because most of the solutions lead to "infinite regress". However, a few minutes of thought makes it clear that learning something new requires prior knowledge to some extent. If one does not know the example to be explained, the example can not be given; if one does not know language, explaining through language will be impossible. Like this, to learning something, one has to have some knowledge before. Then, how can one get the "prior knowledge"?

Plato's answer to this question is "anamnesis"—that is, not knowing means forgetting what one used to know in a previous life and therefore, learning is just recollecting it. In conclusion, Plato's answer suggests that if one does not know something, he is never able to learn it. In other words, structure(or essence) of knowledge or form of cognition independently exists from the beginning. This can be seen as an objectivistic and realistic point of view that epistemology before Kant has. The reason why this attitude is thought objectivistic is that it regards human cognition as a copy of "an object which is formed and defined by itself" and, thus, emphasizes the predominance of the object over the subject of cognition. According to the objectivism, knowledge is just to find or represent the objective and ultimate entity and the knowledge of subject of cognition is defined by this objective entity.

However, the development of modern natural science made it clearer that natural science is not the discovery of objective nature but a way of interpreting natural phenomena. According to Kuhn(1970) who explains the progress in science with the term of "paradigm", normal science is governed and given authority by a paradigm but when a new competitive paradigm is brought in, problems arise—that is to say, the existing paradigm and the new one see the world in each different way and describe it in each different language; within a new paradigm, a concept can have totally different meanings from the ones within the old paradigm, because the new paradigm changes its relation with other concepts. In this regard, progress in knowledge is made not by accumulation of fixed entities but by qualitative changes and requires dialectical reconstruction which integrates the old paradigm and the new one in a comprehensive frame. And the whole frame of the constructed knowledge can be called "structure". What is important here is that the structure is based on human activity which can be considered before distinguishing between a posteriori "matter" and a priori "form" that modern epistemology tries to take as an objective foundation. Therefore, the authority(or validity) of knowledge is autonomously controlled by the activities of those who pursue the knowledge.

This argument that human cognition can not be explained on an objective foundation is in some accordance with Piaget's genetic epistemology. Piaget expounds his operational constructivism can be reduced neither to empiricism nor to apriorism as follows.

> In conclusion, the operational constructivism suggested by genetic analysis is reduced neither to empiricism nor to apriorism, because we could not derive intelligence itself from objects and because the subject does not possess frameworks which contain all reason in advance, but only a certain activity which allows him to construct operational

structures. (Beth & Piaget, 1966, p. 285)

Here, Piaget argues two important points; one is that structure of cognition is constructed and not given a priori; the other is that the construction is operated by human activity. This shows that Piaget's epistemology inherits the construction of the subject, Kant's argument about cognition, and at the same time transcends Kant's idea that the form of sense and understanding is given a priori and, furthermore, advances to genetic argument about human cognition.

Now, let's examine the meaning of structure more.

Mathematical structure

Mathematics has played a key role in the development of structuralism. Structural model of Lévi-Strauss, the father of modern socio-cultural anthropology, is a direct application of algebra and the structure which was first studied in the history is Galois' group(Piaget, 1968). Moreover, after Klein's Erlangen program successfully integrated the traditional algebra and geometry, which exist independently, into a structure called a transformation group, Bourbaki tried to cope with compartmentalization of mathematics, which was divided into geometry, number theory, algebra, analysis, probability calculus and so on, by generalizing basic structures which can include the group structure, and consequently found three matrix structures that can no more be reduced to other structural sources: the first is algebraic structure including group, ring, field etc; the second is order structure like lattice; the third is topological structure dealing with the concepts of neighborhood and continuity.

From these three characterized matrix structures, new structures can derive through combination or differentiation. For example, because a certain set can have two matrix structures at once(combination), there can exist algebraic topology which covers algebraic structures and topological structures. And each matrix structure can strengthen its structure by adding supplementary conditions(differentiation)—for example, as in "group>ring>field", a structure can become more elaborate. De-differentiation, the opposite of differentiation, can be thought of as a process of weakening a structure by excluding a determinative condition—for example, by eliminating the condition of existence of inverse and identity from the group structure, it becomes the semigroup structure.

Now what we have to note here is that Bourbaki's three matrix structures correspond to children's structure of logico-mathematical thought analysed by Piaget(Beth & Piaget, 1966). Piaget understands the nature of thought as operation and refers to an integrated structure made up of operations as grouping. According to him, the structures of concrete operations are explained by 9 groupings in terms of class or relation(Lee, 1973). Operations like classification or seriation constitute the groupings.[1)]

Reversibility is an essential characteristic which draws the line between operations and ordinary actions and therefore, operations can be categorized according to which form of reversibility they have. In short, reversibility can be seen as the permanent possibility that a operation will be able to return to its starting point(Lee, 1973). The following example illustrates other forms of reversibility. Given a balance at equilibrium, when an object is placed on one of its sides, it lose the equilibrium. Equilibrium is restored either by removing the object or by placing an equivalent weight on the other side of the balance; the former is called inversion or negation; the latter is called reciprocity or compensation. Generally, in operational systems related to a set, like classification, reversibility is in the form of inversion; in operational systems related to asymmetry, like seriation, it is in the form of reciprocity.

In terms of the inclusion relation of the set, $A+A'=B$ is adding A' to A and $B+(-A')=A$ is subtracting A' from B. Consequently, the inversion of adding A' and then subtracting A' can be represented as $(+A')+(-A')=0$. As for Bourbaki's three matrix structures, we can find that the algebraic structures have the same form of reversibility. If T is an operation and T^{-1} is its inverse operation, the relation of $TT^{-1}=e$ is common to all algebraic structures.

On the other hand, order structures are characterized by the fact that reversibility in these structures appears in the form of reciprocity: by adding $B\leq A$ to $A\leq B$, getting $A=B$. This is the same structure as of the example of a balance; that is, when an object is placed on one of side of a balance and hence asymmetry of 'order' occurs, equilibrium is restored by putting an equivalent weight on the other side.

Piaget classifies operations into three categories according to whether the reversibility of the operations is in the form of inversion, reciprocity or continuity and separation, and argues that these categories exactly correspond to Boubaki's three matrix structures. According to Piaget(Piaget, 1968), algebraic structures correspond to classification and number; order structures to sequential arrangement, sequential correspondence and seriation; topological structures to operations making classes which are distinguished not by similarities or differences but by the

1) To understand the viewpoint that the structure of thought forms groups or groupings, see Kim, Park & Woo(1984), pp. 134-143.

concepts such as neighborhood, continuity and limit.

Then, what does the understanding about the mathematical structure and the structure of mathematical thought suggest? The modernization movement of mathematics education which embodies Bruner's idea that structure should be taught attempted to present Bourbaki's basic structures directly to children and Piaget's analysis was believed to provide psychological validity for such attempt. What this belief overlooked, however, is that children's thinking in a certain form of structure does not directly assure the intentionalization of the structure in which they are thinking. Piaget does not suggest that Bourbaki's structures can be directly taught to children. The core of Piaget's genetic epistemology is that construction made by reflective abstraction[2] is necessary in order for the learner to absorb an algebraic structure like the concept of group(Hong, 1999). Therefore, from the viewpoint that "structure should be taught", the question not only about mathematical structure but also about the genetic nature of the structure should be at the center of the discussion.

Structure and Genesis

This chapter will analyse the meaning of structure and then examine how the question of genesis is related to the meaning of structure. Let's see Piaget's definition of structure first.

> In short, the notion of structure is comprised of three key ideas: the idea of wholeness, the idea of transformation, and the idea of self-regulation (Piaget, 1968, p. 5).

First of all, it is natural that wholeness is one of the key characteristics of structure, in that the notion of structure is in marked contrast with 'a mere aggregate of parts'. For instance, a symphony is not just a collection of notes but a whole of interrelated constituents. In the same way, the nature of integer does not belong to each integer but instead is relevant to structural properties integrated as a whole (such as group and ring); the structural properties are not related to the properties of each number.

Then, how do the structural properties irrelevant to each component "originate"? When

2) Piaget refers to abstraction of deriving knowledge from qualities of an object as abstraction empirique(empirical abstraction) and abstraction made by general coordination of actions of the subject as abstraction réfléchissante(reflective abstraction).

interpreted in the light of epistemology, this problem becomes similar to Meno's paradox. When a child constructs cognition, does he or she have an innate tool for assimilating each constituent or do the constituents get together and then the structural properties just originate? Although structuralism focuses on the wholeness as mentioned above, the structuralism can be said to regress to metempirical platonism if its viewpoint on the genesis of structure is related to the former question; the structuralism can be said to regress to the combination of empiricism and atomism if it is related to the latter one. Piaget thinks that neither of viewpoints give the satisfactory solution to the problem and presents a new standpoint—operational structuralism. It suggests that structural properties neither exist a priori nor stem directly from constituents but come from relation among constituents, or "operation".

From this point of view, transformation becomes a key characteristic of structure. In fact, every structure can be seen as a system of transformation. In the algebraic structure of natural number, three plus two making five or four following three is an example of transformation and operation is the rule of the transformation. As structuralism becomes more elaborate, it approaches the idea of transformation more closely; the idea of transformation is not found in the theory of Saussure, the father of structuralism but later it becomes a key idea of Chomsky's "transformational-generative grammar". In the case of Bruner, he interprets structure as a fixed entity from the viewpoint of realism and fails to understand structure as systems of transformation for he argues that scholars' knowledge and children's knowledge share the same structure and the differences between the two come from only the way of expression(Hong, 1999).

One of the central ideas of Piaget's theory is that the system of transformation and operation develops. Because children below six usually do not have reversibility in activities such as classification or seriation, they can be said to be at the semi-logical stage of intellectual development in that they lack half of the logic of inverse operation(Piaget, 1968). Nevertheless, the concepts of function and identity —even though they are rudimentary— exist at this stage. In concrete operational period when children acquire reversibility, the concept of number is constructed by transitivity of relation, quantification of classification, seriation and inclusion relation and thereby the system of operation gains the semigroup structure. In formal operational period when inversion and reciprocity, two forms of reversibility, are coordinated, the system of operation gains INRC group[3] or Klein 4-group.

Now, the only problem is for what reason this structure of operation is able to develop. Piaget

3) When I stands for Identity; N for negation; R for reciprocity; C for correlation between negation and reciprocity, these satisfy $I^2=N^2=R^2=C^2=I$, $NRC=I$, $NR=C$, $RC=N$, $CN=R$.

answers that the reason for the occurrence of reversibility is that structure is constantly regulated by the requirement for equilibrium. The concept of self-regulation is related to this; that is, though structure and transformation always coexist, transformation can not go beyond the scope of cognition at once but should maintain invariant bounds to some extent. And the invariant bounds are established when the condition that operations are always able to return to the starting point by inversion functions as a rule of general coordination of operations. This illustrates that reversibility take an important place in the discussion about structure.

Bruner's structure

The attempt to derive the structure of a subject from educational practice and create teaching-learning theories of it is also closely related to the meaning of structure which has been studied so far. Therefore, the meaning of structure that Bruner —who argues on the basis of Piaget's theory of intellectual development that the structure of knowledge should be taught— explains by emphasizing basic ideas, principles and laws needs to be reinforced. In this regard, suggesting that what Bruner means by the structure of knowledge should be reinterpreted, Park(1981) says: "the structure of knowledge is not the fragmented topics on the surface level of knowledge but the principles inherent in it. The principles should incessantly have relation to the surface level and, at the same time, to other general principles, the constituents of structure, and the relation must be controlled by the rules of transformation."

The structure of knowledge explained by Bruner, however, does not reflect the abundant meanings of structure that are constantly created and reconstructed, and shows differences in viewpoint from Piaget, which seem to come from the fundamental differences in the standpoint toward epistemology. The typical example is Bruner's argument that a child aged eight can be taught quadratic equations or, as mentioned above, that the scholars at the forefront of a discipline and the students on the elementary level can share the same structure of knowledge. This idea that structure of knowledge is the same to everyone shows that Bruner's concept of structure is not different from an objectivistic and realistic concept of it, which is also found in Plato's anamnesis. Plato suggests that a concept is in a one-to-one correspondence with the experience from which the concept is derived and it is a fixed tool for interpreting phenomena. Bruner says "to understand structure means to have learned a model for understanding other things like it that one may encounter" and "these basic ideas should be isolated and taught more explicitly in a manner that frees them from specific areas(Bruner, 1960)." But the problem of this explanation is that the knowledge here is explained in the light of atomism. Especially in the case of mathematics and science, knowledge is a system of concepts rather than a gathering of individual facts and information. In the same context, Piaget stresses the wholeness as one of the characteristics of structure. A separate fact does not bear a fixed meaning and its meaning varies according to its context. Bruner himself says it is undesirable to teach just piles of facts, but a separate basic idea may also be reduced to just a fact if it loses its whole context.

Besides, though Bruner's intention can be understood as emphasizing structure as a system or interrelation among concepts, the problem that scholars' knowledge and children's knowledge should have the same structure is still unsolved. The differences between professional scholars' knowledge and children's knowledge come neither from differences in quantity nor from differences in the methods of representation. As Piaget says in his explanation of the generalization by reflective abstraction(Piaget et al., 1977), the differences in the structures of different levels contain qualitative gaps. In addition, as Piaget explains with the phrase of "self-regulating transformation", cognitive structure and, further, the structure of knowledge are not being fixed but being incessantly modified toward the ideal form. But Bruner regards structure as a fixed tool for interpreting phenomena. Like this, he fails to define the concept of structure thoroughly and as a result —unlike his intention— his theory regresses to an objectivistic and realistic standpoint.

Representation vs. Operation

According to Piaget, knowledge has an operative aspect and a figurative aspect. The two aspects are also called "schème" and "schéma" respectively; schème means a general structure that makes actions and operations possible and generalizes them; schéma means representation or image of the result from a certain action or operation. Perception, imitation, image and so on, which Piaget calls a figurative aspect can not be the essence of knowledge and they can have meanings only through actions or operations. Especially logico-mathematical knowledge sometimes does not have the figurative aspect[4].

4) A typical example is cardinality of natural numbers. This can be understood only after recognizing

As for the essence of logico-mathematical concepts, Piaget explained as follows: "first, the structure of a biological organism (the starting point) leads to a sensori-motor schème and, then according to it, actions are coordinated, resulting in reflective abstraction, by which operations are constructed, and based on them, higher-level operations are formed; it is the operations and the higher-level operations that are the essence of logico-mathematical concepts"(Kim et al., 1984). From Piaget's viewpoint, logico-mathematical concepts should not be derived from empirical contents but it should be constructed by operational forms and therefore, reflective abstraction is obviously different from empirical abstraction. In addition, although the subject of cognition has to go through représentation(representation) of his actions in order to coordinate them, représentation is needed only at the first stage, where réfléchissement starts, of the whole process of reflective abstraction, which is made up of a cycle of réfléchissement and réflexion(Piaget et al., 1977). The fact that this representation is possible is just an evidence showing that the subject of cognition internalized his actions. Moreover, the réfléchissement is finished only after the subject of cognition makes réflexion start through thématisation of his internalized actions and, thus, it is naturally impossible that the mathematical concepts requiring reflective abstraction are completely explained by only representation.

Bruner's idea of three representation needs to be reconsidered in this regard. His suggestion that the structure of a subject should be translated and then presented to children implies that the teacher should assimilate new knowledge to the student's existing cognitive structure by translating the knowledge into the mode of representation, which is easy for the student to understand and then presenting it. However, this is unacceptable, considering Piaget's genetic epistemology which emphasizes the children's cognitive structure evolves through qualitative changes. As studied above, Piaget's genetic epistemology or Popper's and Kuhn's theory of scientific history shows that the structure of knowledge develops not just through quantitative accumulation but through incessant modifications and transformations. If the nature of knowledge is so, teaching is not translating new knowledge into already known situations for students but making them look upon the already known situations as new tools for interpretation and, in the long run, instructing them to see even the familiar situations in more thought-provoking way. As in Bruner's EIS theory, the educational situation where the learner just accepts the mode of representation presented by the teacher does not reflect that the learner's cognitive structure changes and therefore it cannot be

one-to-one correspondence as a logical operation and bears no relation to concrete objects, themselves, or the spatial arrangement (or a figurative aspect) of them. (refer to p.130 of Kim et al, 1984 for more examples).

said that new learning has actually happened. Between knowing if one move out farther from the center of a see-saw, he or she goes down father and understanding Newton's Law of Moments, there are qualitative differences in cognitive structure and the differences are logical gaps more than just the differences in representation. Nevertheless, if the teacher tries to teach the fundamental principle of Law of Moments using enactive representation such as a see-saw and a balance, the student is likely only to accept the principle superficially on his level at that time.

Moreover, Bruner thinks that if the instructor presents the representation which is higher than the student's stage of development, the challenges which are derived from the gap between the two cause new learning. But from Piaget's point of view, representation is just a product of cognition at its level and at the next level of cognition it works only as a concrete object but it cannot be a source of new cognition. The idea[5] that presenting various kinds of representation makes new cognition happen comes not from the viewpoint that the subject constructs cognition but from the realistic viewpoint that the subject just copies objective objects of cognition.

In the case of logico-mathematical concepts, particularly, their essence is not representation but operation, as Piaget points out. In addition, mathematical structures, concepts, methods of proof, algorithms, propositions, theorems and so on are all schèmes and mathematical activities include all activities of constructing and applying these operational schèmes. The learner can construct new mathematical concepts only through reflective abstraction based on his operations as the objects of cognition and after that he is able to understand representation, presented by the teacher. in his own way.

The learning of algebraic structures

So far we have studied essential characteristics of structure such as wholeness, transformation and self-regulation. Now it is time to consider the situation where the instructor tries to teach algebraic structures to the student. The problem here is how the formalized algebraic structures assimilate to the learner's cognitive structure. For instance, if the reason for teaching the group structure is that it is the most fundamental and essential part of mathematical thought, it is wondered how closely learning the simple fact that "the a set of real numbers has such and such characteristics" approaches to the learning of structure that is originally intended[6]. According to

5) A similar argument is found in Dienes' "perceptual variability principle".

Piaget(1986), reflective abstraction, the mechanism by which structure gets more elaborate and refined, plays a key role not only in the process where logico-mathematical thought acquires the group structure but also in the process where the learner grasps the concept of group as the result of thematizing and intentionalizing the thought. If teaching algebraic structures is necessary, the most important principle of teaching structure should be that algebraic structures should be constructed through the abstraction and the generalization of the learner's way of thinking and mathematical experiences.

The concept of group is a basic concept of modern algebra and moreover since its utility is very comprehensive, it is introduced to almost all fields of mathematics, logics, physics and so forth. Piaget(1968) even says group is the prototype of general structures. And he also explains the reason the concept of group is very comprehensive and successful is that the concept is acquired by reflective abstraction, the special way of logico-mathematical abstraction. When a certain property is abstracted from the things that have the property, the more general the property becomes (the more its extension extends), the narrower its intension becomes and as a result the utility of the structure decreases. On the other hand, when a concept is abstracted from operations(the ways of operating on things), the more generalized the concept is, the richer its intension becomes and thus the utility of the structure becomes more comprehensive.

Meanwhile, because the concept of group is derived through reflective abstraction, or through abstraction of various operations, general coordination rules, that is, basic methods of coordinating operations, should be first analysed in order to analyse the structure of the concept. As examined above, since the system of transformation should be self-regulated within certain bounds so as to have a form of structure, at least two basic rules of coordinating operations are needed as follows(Piaget, 1968): ① An operation must be able to return to its starting point all the time (through inversion), ② The ultimate results of operations must be independent of the chosen alternative course.

In comparison with mathematical definition of the concept of group, the rule ① is related to the existence of inverse element; the rule ② to the associative law. In conclusion, this shows that the group structure is a representative structure that contains essential characteristics of the structure of operations or the structure of mathematical thought.

Since the algebraic structures are fundamentally the transformational system, just presenting the

6) Bruner says that understanding the structure of a subject has the advantage of ①making the subject easy to understand and ② easy to remember, ③ enabling the transition of learning and ④ narrowing the gap between advanced knowledge and elementary knowledge.

ultimate form of formalized structures is not an appropriate way to teach the algebraic structures to the student. Piaget(1968) stresses that "structure" is not correspondent to "form". According to him, the thought of structure as the transformational system is in succession to the thought of construction as a continual process of formation. In this regard, Bruner's so-called descending curriculum which just lowers the level of the formalized structure of knowledge in the order from the high school to the middle school and to the elementary school cannot be said to conform to the viewpoint of structuralism. According to Freudenthal(1991) who defines structure as a tool for organizing mathematical phenomena or mathematics itself, since in order to help the learner find and organize mathematical structures the content and form of mathematics must be dialectically interwoven and levelled up, the instructor should start with the reality of the learner's level and then seek to systemize it into mathematically refined structures instead of translating formalized structural results of mathematics into the language that the learner can understand.

In other words, instead of presenting complete structures of mathematics to the learner, the instructor ought to have the learner face the phenomena that are suitable for the learner's mental reality and where mathematical structures can become a tool for systemization and let the learner experience the process of reinvention where he can find out the tool for systemization by himself. When an instructor let a learner deal with the tools for organization in the phenomena first, the learner constructs the mental objects of mathematical objects and then is expected to attain then a rise in level of understanding through the process of reflection. Similarly, in teaching algebraic structures including group, the instructor should present the phenomena, which will be mathematized, to allow the learner to construct the mental objects of algebraic structures and lead him to turn the structures into mathematical concepts through reflection, without starting with formalized structural concepts and then studying concrete materials of the concepts.

Now, let's examine the examples where the concept of group, one of the most basic algebraic structures, is translated into concrete examples and where the concept of group is constructed from phenomena, and find out differences. In teaching group structure, instructors usually give examples as follows: ① The set of all the symmetries of the rectangular(not the square) forms the group $Z_2 \oplus Z_2$. ② The set of 2×2 matrices $GL_2(R)$, forms a group under multiplication. ③ On the quadratic extension field $Q(\sqrt{2})$, identity map i and $\tau : a+b\sqrt{2} \rightarrow a-b\sqrt{2}$ forms the Galois group of $Q(\sqrt{2})$ over Q.

The fact that the sets in these examples form a group is verified by accurate calculations and the process has no errors. But the problem of this frame is that algorithm is repeated whenever a new group is introduced. To prove whether something constructed forms a group becomes an endlessly repeated simple work for learners, without having the opportunity to discern the genetic nature of group structures, they have to just accept the fact that given sets 'mysteriously' have the same structure.

But we can explain that the meaning of group, which is shared by all the above examples, could be understood as "a system of isomorphisms of a certain structure". From this viewpoint, ① shows the set of congruent transformations preserving the structure of a figure forms the group. In ②, $GL_2(R)$ is the set of linear transformations that send a line passing through the origin to a line passing through the origin on the plane. In ③, $\tau: a+b\sqrt{2} \rightarrow a-b\sqrt{2}$ is an isomorphism preserving the structure of addition and multiplication that is defined in $Q(\sqrt{2})$.

The biggest advantage of introducing the group as isomorphisms is that the fact that what is defined is the group is guaranteed by the way of introduction itself. If we designate the set of all the isomorphisms of the structure S as G, identity map obviously belongs to G; if f is an element of G, the reverse of it naturally belongs to G; if f and g are the elements of G, $f \circ g$ also naturally belongs to G. Therefore, that "isomorphism of a certain structure forms a group with the operation of composition" can be the starting point of the concept of group. Moreover, this is the conceptual approach, which is much more explicit than algorithmic proof and the conceptual approach is a characteristic that modern mathematics is aiming at.

Conclusion

Bruner's idea is usually called "curriculum of structuralism". What is important here, however, is not the term 'structure' but what he really means by structure. Bruner says that the subjects (necessary to study science, like number, measurement and probability) should be taught in intellectually right form in a manner appropriate to children's way of thinking from the very early age and the subjects must be repeatedly covered in higher years. As in this argument, putting basic ideas of knowledge in textbooks from the lowest year and constantly repeating them even in higher years is likely to be thought as teaching structure. Park(1981) explains that, in order to make Bruner's ideas of structure and spiral curriculum meaningful, the contents of subjects should be regarded as what scholars do or the process of study. However, from the viewpoint of subject matter education, which is closely related to the contents of knowledge, such explanation is just indefinite. Bruner himself suggests EIS theory in an attempt to apply the idea of spiral curriculum to the concrete contents of subjects, but it reveals many theoretical limits.

Piaget's operational constructivism emphasizes that the child ought to construct knowledge through his own activities and, especially, the source of logico-mathematical cognition should be operations or internalized actions. On the other hand, Bruner argues that the structure of knowledge is a fixed entity and the instructor should translate it into representation appropriate to the children's cognitive structure. Regardless of its intent, Bruner's argument seems to highlight the functional side of education or the educational method of transmitting given knowledge quickly and efficiently and, when knowledge is translated, its intellectual honesty is likely to be damaged.

The idea that the instructor should establish general ideas and principles of mathematics and level down them to the learners' level implies that the structure of mathematics is completed and it is a fixed intellectual system. However, in the view of structuralism including Piaget's, structure is the transformational system which is developing and changing. If this gets involved with the contents of subjects, the learner's developing structure of knowledge entails the qualitative differences in the level, and the mechanism of reflective abstraction like "content→form→more refined content→new form→..." becomes a key part which constitutes the structure.

Freudenthal argues that to help the learner understand the structure of mathematics, the learner needs to experience the structuralization, through which he organizes phenomena into mathematical structures. The direction of the structuralization or the organization is directly opposite to knowing the formal meaning of a certain structure and dealing with a few phenomena with the structure as examples. In order to realize the premise that the mathematical structure should be learned not as what is given to the learner but as what the learner himself should construct, it is necessary to research more intensively on the nature of mathematical structure and the phenomena in the real world, which can be mathematized into the structure.

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초 록

수학적 구조의 의미에 대한 교육학적 고찰

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본 연구는 삐아제의 발생적 인식론과 구조주의에 대한 고찰을 바탕으로, 부르바키 학파가 주장하는 수학의 모구조와 삐아제가 말하는 수학적 사고의 구조가 교육적 관점에서 어떻게 관련을 맺는지 살펴보았고, 지식의 구조가 가져야 하는 본질적인 특성을 검토하면서 변형과 발달이 핵심적인 문제가 됨을 지적하였다. 이에 비추어 브루너의 구조 개념이 갖는 한계를 지적하였으며, 수학적 개념으로서의 대수적 구조를 학습자가 구성하게 하는 문제를 논의하였다.

주제어 : 구조주의, 발생적 인식론, 구조